A regularity theory for multiple-valued Dirichlet minimizing maps *

Wei Zhu

February 2, 2008

Abstract

This paper discusses the regularity of multiple-valued Dirichlet minimizing maps into the sphere. It shows that even at branched point, as long as the normalized energy is small enough, we have the energy decay estimate. Combined with the previous work by Chun-Chi Lin, we get our first estimate that $\mathcal{H}^{m-2}(\text{singular set}) = 0$. Furthermore, by looking at the tangent map and using dimension reduction argument, we show that the singular set is at least of codimension 3.

2000 Mathematics Subject Classification: Primary 49Q20

*Department of mathematics, Rice University, Houston, TX 77005; weizhu@math.rice.edu

Contents

1 Introduction

The regularity of harmonic maps between Riemannian manifolds has been a fascinating subject in recent years. The very first general result on this is due to [SU1], in which they proved that a bounded, energy minimizing map $u : M^n \to N^k$ is regular (in the interior) except for a closed set S of Hausdorff dimension at most $n-3$. One important technique they use in the paper is for lowering the dimension of S under the condition that certain smooth harmonic maps of spheres into N are trivial. This can be checked in some interesting cases, for example if N has nonpositive curvature. They showed $S = \emptyset$, i.e, any energy minimizing map into such a manifold is smooth. Use that method, they [SU2] are also able to reduce the dimension of S if N is a sphere. The result is as follows:

Theorem 1.1 ([SU2], Theorem 2.7). *For $k \geq 2$, define a number $d(k)$ by setting*

$$d(2) = 2, d(3) = 3$$

$$d(k) = [\min\{\frac{k}{2} + 1, 6\}] \text{ for } k \geq 4$$

where $[\cdot]$ denotes the greatest integer in a number. If $n \leq d(k)$, then every energy minimizing map from a manifold M of dimension n into $\mathbb{S}^k \subset \mathbb{R}^{k+1}$ is smooth in the interior of M. If $n = d(k)+1$, such a map has at most isolated singularities, and in general the singular set is a closed set of Hausdorff dimension at most $n - d(k) - 1$.

This same question in liquid crystal configurations setting($n = 3, k = 2$) has been studied independently by Hardt-Kinderlehrer-Lin using blowing-up argument in [HKL].
A few years later, Theorem 1.1 was extended to stable-stationary harmonic maps $u \in H^1(\Omega, \mathbb{S}^k)$, $k \geq 3$ by Hong-Wang [HW]. Stable-stationary harmonic maps are harmonic maps with zero domain first variation and nonnegative range second variation. Examples of stable-stationary harmonic maps include energy minimizing maps.
In a recent work of Lin-Wang [LW], they improved the theorems by [SU2], [HW] for $4 \leq k \leq 7$ as follows:

Theorem 1.2 ([LW], Theorem 1). *Define*

$$\tilde{d}(k) = \begin{cases} 3 & k = 3 \\ 4 & k = 4 \\ 5 & 5 \leq k \leq 9 \\ 6 & k \geq 10. \end{cases}$$

For $k \geq 3$, let $u \in H^1(\Omega, \mathbb{S}^k)$ be a stable-stationary harmonic map, then the singular set S of u has Hausdorff dimension at most $n - \tilde{d}(k) - 1$.

We can also talk about the energy minimizing problems in the setting of multiple-valued functions (maps) thanks to the monumental work [AF]. After Almgren gave suitable definitions of derivative and Sobolev space for multiple-valued functions, the question of minimizing energy among functions with the prescribed boundary data becomes legitimate. Furthermore, he was able to show that any Dirichlet energy minimizing multiple-valued function is regular in the interior and has branch point of codimension at least 2. Although the primary purpose in [AF] of introducing multiple-valued functions is to approximate almost flat mass-minimizing integral currents by graphs of Dirichlet minimizing multiple-valued functions, the subject of multiple-valued functions in the sense of Almgren turns out to be also interesting in its own. See some recent works [CS], [GJ], [LC1], [LC2], [MP], [ZW1], [ZW2], [ZW3].
In the same spirit of [AF], Chun-Chi Lin (in [LC1]) considered the energy minimizing multiple-valued map into spheres. Specifically, he showed that for points not in the branch set B_0, as long as the normalized energy is small, the map is regular there (see more of this discussion in section three). We will continue his work by examining the local behavior of points in B_0. The main idea is to use the blowing-up analysis at this point. The blowing-up sequence converges strongly to a Dirichlet minimizing function which is regular due to [AF]. Hence it guarantees the energy of the original map near this point satisfies some growth condition. Combining our result with the result in [LC1], we conclude that the minimizing map is regular at any point as long as the normalized energy there is small enough thanks to Morrey's growth lemma. This gives us the first $m-2$ estimate. Then, using dimension reduction argument, we get our main result:

Theorem 1.3. *Let $u \in \mathcal{Y}_2(\mathbb{B}_1^m(0), \underline{Q}(\mathbb{S}^{n-1}))$ be a strictly defined, Dirichlet minimizing map. Then it is Hölder continuous away from the boundary except for a closed subset $S \subset \mathbb{B}_1^m(0)$ such that $\dim(S) \leq m-3$.*

The assumption that we are looking at points in B_0 is important in the blowing-up process because we need to get suitable constant of the form $Q[[b]]$ for some $b \in \mathbb{S}^{n-1}$ in order for the subtraction between two Q-tuples to make sense.
There are some other interesting questions which are not addressed in this paper, and still open to the author's knowledge. A first one is whether our result is an optimal one. We are hoping to have some similar results as in [SU2], [LW]. Some new techniques are expected because [SU2][LW] both use Bochner formula, which is no longer available in the multiple-valued functions setting.
A second one is the regularity for stationary harmonic multiple-valued functions. There are already some positive results for this in the two dimensional case, see [LC2].
Another one is the branching behavior. Chun-Chi Lin (in [LC1]) has done some work on this. But there was some problem with that. Basically speaking, the monotonicity formula for frequency function he used in his proof actually does not necessarily hold for multiple-valued maps. Some new idea is probably needed to get around this obstacle.
It is my great pleasure to thank my thesis advisor Professor Robert Hardt for

his support, encouragement and kindness during the years at Rice. A lot of this work was stimulated by [HKL].

2 Preliminaries

Most of the notations, definitions and known results about multiple-valued functions that we need can be found in [ZW1]. The reader is also referred to [AF] for more details. We also use standard terminology in geometric measure theory, all of which can be found on page 669-671 of the treatise *Geometric Measure Theory* by H. Federer [FH].
For reader's convenience, here we state some useful results not included in [ZW1]. The proofs of them can be found in [AF].

Theorem 2.1 ([AF], §2.6). *(a)* $0 < r_0 < \infty$.
(b) $A \subset \mathbb{R}^m$ *is connected, open, and bounded with* $\mathbb{B}^m_{r_0}(0) \subset A$. ∂A *is an* $m-1$ *dimensional submanifold of* $\mathbb{R}^m$ *of class 1.*
(c) $f : A \to \mathbb{Q}$ *is strictly defined and is Dir minimizing.*
(d) $D, H, N : (0, r_0) \to \mathbb{R}$ *are defined for* $0 < r < r_0$ *by setting*

$$D(r) = Dir(f; \mathbb{B}^m_r(0))$$
$$H(r) = \int_{\partial \mathbb{B}^m_r(0)} \mathcal{G}(f(x), Q[[0]])^2 d\mathcal{H}^{m-1}x$$
$$N(r) = rD(r)/H(r) \text{ provided } H(r) > 0.$$

(e) $\mathcal{N} : A \to \mathbb{R}$ *is defined for* $x \in A$ *by setting*

$$\mathcal{N}(x) = \lim_{r \downarrow 0} r Dir(f; \mathbb{B}^m_r(x)) / \int_{\partial \mathbb{B}^m_r(x)} \mathcal{G}(f(z), Q[[0]])^2 d\mathcal{H}^{m-1}z$$

provided this limit exists.
(f) $H(r) > 0$ *for some* $0 < r < r_0$.
Conclusions.
For $\mathcal{L}^1$ *almost all* $0 < r < r_0$,
Squash Deformation:

$$D(r) = \int_{x \in \partial \mathbb{B}^m_r(0)} < \xi_0 \circ f(x), D(\xi_0 \circ f)(x, x/|x|) > d\mathcal{H}^{m-1}x$$

Squeeze Deformation:

$$r \cdot D'(r) = (m-2)D(r) + 2r \int_{x \in \partial \mathbb{B}^m_r(0)} |D(\xi_0 \circ f)(x, x/|x|)|^2 d\mathcal{H}^{m-1}x$$

Remark 2.1. *(1) For convenience, we will use* $\partial f / \partial r$ *to represent* $Df(x, x/|x|)$ *for any multiple-valued function* f *whenever the derivative exits.*
(2) Noticing that the squeeze deformation comes from a domain deformation, the squeeze deformation formula still holds for multiple-valued maps.

(3) We can replace those ξ_0 by ξ because that in the proof of those formulas, the only things we need for ξ_0 are that $\xi_0(\lambda x) = \lambda\xi_0(x)$, for $\lambda > 0$, $x \in \mathbb{Q}$, and $D(r) = \int_{B_r^m(0)} |D(\xi_0 \circ f)|^2 d\mathcal{L}^m$, both of which still hold for ξ.

Theorem 2.2 ([**AF**], §2.9). *Corresponding to numbers $0 < s_0 < \infty$, $1 < K < \infty$, and(not necessarily distinct points) $q_1, \cdots, q_Q \in \mathbb{R}^n$ we can find $J \in \{1, \cdots, Q\}, k_1, \cdots, k_J \in \{1, \cdots, Q\}$, distinct points $p_1, \cdots, p_J \in \{q_1, \cdots, q_Q\}$, and $s_0 \le r \le Cs_0$ such that*
(1) $|p_i - p_j| > 2Kr$ for each $1 \le i < j \le J$,
(2) $\mathcal{G}(\sum_{i=1}^{Q}[[q_i]], \sum_{i=1}^{J} k_i[[p_i]]) \le Cs_0/(Q-1)^{1/2}$,
(3) $z \in \mathbb{Q}$ with $\mathcal{G}(z, \sum_{i=1}^{Q}[[q_i]]) \le s_0$ implies $\mathcal{G}(z, \sum_{i=1}^{J} k_i[[p_i]]) \le r$,
(4) in case $J = 1$, diam(spt($\sum_{i=1}^{Q}[[q_i]]$)) $\le Cs_0/(Q-1)$; here

$$C = 1 + [(2K)(Q-1)^2]^1 + [(2K)(Q-1)^2]^2 + \cdots + [(2K)(Q-1)^2]^{Q-1}.$$

Theorem 2.3 ([**AF**], §2.10). *Corresponding to*
(a) $J \in \{1, 2, \cdots, Q\}$,
(b) $k_1, k_2, \cdots, k_J \in \{1, 2, \cdots, Q\}$ with $k_1 + k_2 + \cdots + k_J = Q$,
(c) distinct points $p_1, p_2, \cdots, p_J \in \mathbb{R}^n$,
(d) $0 < s_1 < s_2 = 2^{-1} \inf\{|p_i - p_j| : 1 \le i < j \le J\}$,
we set

$$\mathbb{P} = \mathbb{Q} \cap \{\sum_{i=1}^{Q}[[q_i]] : q_1, \cdots, q_Q \in \mathbb{R}^n \text{ with } card\{i : q_i \in \mathbb{B}_{s_1}^n(p_j)\} = k_j \text{ for each } j = 1, \cdots, J\}$$

Conclusions*:*
There is a map $\Phi : \mathbb{Q} \to \mathbb{P}$ such that
(1) $\Phi(q) = q$ whenever $q \in \mathbb{Q}$ with $\mathcal{G}(q, \sum_{i=1}^{J} k_i[[p_i]]) \le s_1$,
(2) $\Phi(q) = \sum_{j=1}^{J} k_j[[p_j]]$ whenever $q \in \mathbb{Q}$ with $\mathcal{G}(q, \sum_{i=1}^{J} k_i[[p_i]]) \ge s_2$,
(3) $\mathcal{G}(q, \Phi(q)) \le \mathcal{G}(q, \sum_{i=1}^{J} k_i[[p_i]])$ for each $q \in \mathbb{Q}$,
(4) Lip $\Phi \le 1 + Q^{1/2}s_1/(s_2 - s_1)$.

Theorem 2.4 ([**AF**], §2.13). *(a) In case $m = 2$, $\omega_{2.13} = 1/Q$.*
(b) In case $m \ge 3, 0 < \epsilon_Q < 1$ is as defined as in [AF], §2.11 and $0 < \omega_{2.13} < 1$ is defined by the requirement

$$m - 2 + 2\omega_{2.13} = (m-2)(1+\epsilon_Q)/(1-\epsilon_Q).$$

(c)

$$\Gamma_{2.13} = 4^{1-\omega_{2.13}}[2^{m/2}/(1-2^{-\omega_{2.13}}) + 3 \cdot 2^{m-1+\omega_{2.13}}](m\alpha(m))^{-1/2} Lip(\xi)^2 Lip(\xi^{-1}).$$

(d) $f \in \mathcal{Y}_2(\mathbb{R}^m, \mathbb{Q})$ is strictly defined and $f|\mathbb{B}_1^m(0)$ is Dir minimizing with $Dir(f; \mathbb{B}_1^m(0)) > 0$.
Conclusions.
(1) For each $z \in \mathbb{B}_1^m(0), 0 < r < 1 - |z|$, and $0 < s \le 1$,

$$Dir(f; \mathbb{B}_{sr}^m(z)) \le s^{m-2+2\omega_{2.13}} Dir(f; \mathbb{B}_r^m(z)).$$

(2) Whenever $0 < \delta < 1$ *and* $p, q \in \mathbb{B}^m_{1-\delta}(0)$,

$$\mathcal{G}(f(p), f(q)) \leq \Gamma_{2.13}\delta^{-m/2} Dir(f; \mathbb{B}^m_1(0))^{1/2}|p-q|^{\omega_{2.13}},$$

in particular, $f|\mathbb{B}^m_{1-\delta}(0)$ *is Hölder continuous with exponent* $\omega_{2.13}$.
(3) Corresponding to each bounded open set A *such that* ∂A *is a compact* $m-1$ *dimensional submanifold of* $\mathbb{R}^m$ *of class 1, there is a constant* $0 < \Gamma_A < \infty$ *with the following property. Whenever* $g \in \mathcal{Y}_2(A, \mathbb{Q})$ *is Dir minimizing and* $p, q \in A$,

$$\mathcal{G}(g(p), g(q)) \leq \Gamma_A Dir(g; A)^{1/2} \sup\{dist(p, \partial A)^{-m/2}, dist(q, \partial A)^{-m/2}\}|p-q|^{\omega_{2.13}}.$$

3 Some Remarks on [LC1]

In [LC1], Chun-Chi Lin introduced the set
$B_0 = \{x \in \mathbb{B}^m_1(0)$: a Lebesgue point of $f, \xi^{-1} \circ \rho \circ AV_{r,x}(\xi \circ f) = Q[[b_r]]$,
for any small enough radius $r > 0, b_r \in \mathbb{R}^n$ and $AV_{r,x}(\xi \circ f) = \fint_{\partial \mathbb{B}^m_r(x)} \xi \circ f\}$.
He proved that for a point not in B_0, if the normalized energy of f is small enough there, then the energy of f near this point satisfies some growth condition. The key ingredients are the induction on Q and finding a comparison map. In order to use the induction, we need $J \geq 2$. This is guaranteed by our assumption that the point we are looking at is not in B_0. He did not explain that in his paper. Here is the detail:
If $a \notin B_0$, i.e. there is $r > 0$, such that $\xi^{-1} \circ \rho \circ AV_{r,a}(\xi \circ f) \neq Q[[b]]$ for any $b \in \mathbb{R}^n$. We may as well just assume that

$$\xi^{-1} \circ \rho \circ AV_{1,a}(\xi \circ f) \neq Q[[b]] \text{ for any } b \in \mathbb{R}^n.$$

Now instead of letting $q^* \in \mathbb{Q}^*$ be the point in $\mathbb{Q}^*$ such that

$$|q^* - AV_{1,a}(\xi \circ f)| = \text{dist}(AV_{1,a}(\xi \circ f), \mathbb{Q}^*)$$

we let $q^* = \rho \circ AV_{1,a}(\xi \circ f)$, and $q = \sum_{i=1}^Q [[q_i]] = \xi^{-1}(q^*)$. With these points $q_1, q_2, \cdots, q_Q$, $1 < K < \infty$ and constant s_0 to be chosen later, we find $J \in \{1, 2, \cdots, Q\}$, $k_1, k_2, \cdots, k_J \in \{1, 2, \cdots, Q\}$, distinct points $p_1, \cdots, p_J \in \mathbb{R}^n$ as in Theorem 2.2. Let $q_0 = \sum_{i=1}^J k_i[[p_i]]$.
If $J = 1$, from Theorem 2.2 (4)

$$\begin{aligned}\text{diam}(\text{spt}(\sum_{i=1}^Q [[q_i]])) &= \text{diam}(\text{spt}(\xi^{-1} \circ \rho \circ AV_{1,a}(\xi \circ f))) \\ &\leq Cs_0/(Q-1);\end{aligned}$$

but we already know that $\xi^{-1} \circ \rho \circ AV_{1,a}(\xi \circ f) \neq Q[[b]]$ for any $b \in \mathbb{R}^n$, hence $\text{diam}(\text{spt}(\sum_{i=1}^Q [[q_i]]))$ is a fixed positive number. So we can choose s_0 small enough to guarantee that $J \geq 2$.
We also have to show that the rest of the proof in [LC1] is still valid after we

choose the different q^*. This is because the only place where q^* is used in [LC1] is to show

$$\int_{\partial\mathbb{B}_1^m(a)} |\xi\circ f(x)-q^*|^2 \le C\int_{\partial\mathbb{B}_1^m(a)} |\xi\circ f(x)-AV_{1,a}(\xi\circ f)|^2 \text{ for some constant } C.$$

We still have this because
$\int_{\partial\mathbb{B}_1^m(a)} |\xi\circ f - q^*|^2 = \int_{\partial\mathbb{B}_1^m(a)} |\xi\circ f(x) - \rho\circ AV_{1,a}(\xi\circ f)|^2$
$= \int_{\partial\mathbb{B}_1^m(a)} |\rho\circ\xi\circ f(x)-\rho\circ AV_{1,a}(\xi\circ f)|^2 \le (\text{Lip }\rho)^2 \int_{\partial\mathbb{B}_1^m(a)} |\xi\circ f(x)-AV_{1,a}(\xi\circ f)|^2$.
Another thing that worth mentioning is in the proof of Lemma 4 in [LC1], more precisely (2.12). He was claiming that g_j is Hölder continuous hence having growth condition on the energy. But in fact since his work is only on points outside B_0, and we do not know whether the origin is inside or outside of the set B_0 for each g_j, the induction seems to be a problem. However, using our result on branched points, we can overcome this. Let's look at our result Theorem 7.1 in advance (notice that our proof does not depend on induction or the result in [LC1]) , which says that at branched point,

$$D(r) \le Cr^{m-2+\omega_{2.13}},$$

for some constant C depending on the dimensions and the total energy $D(1)$. Now we claim that for each g_j in [LC1], there exists a positive constant α such that

$$D_g(r(1-t_Q)) = \sum_{j=1}^{J} D_{g_j}(r(1-t_Q)) \le C(\alpha, m, n, Q, \text{total energy of f})r^{m-2+\alpha}.$$

This is because if the origin is not in the corresponding set B_0 of g_j, then the induction argument gives us the above estimate. Otherwise, our result applies. Finally, we modify the end of proof of Lemma 4 in [LC1] as following: (reason that the original proof did not work is that by considering two cases, the integration did not necessarily work)
Now we have

$$D_f(r) \le \frac{8}{7}Cr^{m-2+\alpha} + \frac{1}{7(m-1)}rD_f'(r).$$

Let's denote $D_f(r)$ by $\phi(r)$. The original inequality becomes

$$\phi(r)' - \frac{\phi(r)}{r}7(m-1) + 8Cr^{m-3+\alpha}(m-1) \ge 0.$$

Multiply both sides by $r^{-7(m-1)}$, we get

$$\frac{d}{dr}[\phi(r)r^{-7(m-1)} + \frac{8C(m-1)}{5-6m+\alpha}r^{5-6m+\alpha}] \ge 0.$$

Hence

$$\phi(r)r^{-7(m-1)} + \frac{8C(m-1)}{5-6m+\alpha}r^{5-6m+\alpha} \le \phi(1) + \frac{8C(m-1)}{5-6m+\alpha} := M,$$

$$
\begin{aligned}
D_f(r) = \phi(r) &\leq (M + \frac{8C(m-1)}{6m-5-\alpha} r^{5-6m+\alpha}) r^{7(m-1)} \\
&= M r^{7(m-1)} + \frac{8C(m-1)}{6m-5-\alpha} r^{m-2+\alpha} \\
&\leq \max(M, \frac{8C(m-1)}{6m-5-\alpha}) r^{m-2+\alpha}
\end{aligned}
$$

while the last inequality follows because $7(m-1) > m-2+\alpha$.

4 Maximum Principle for Multiple-Valued Dirichlet Minimizing Functions

Lemma 4.1. *Given a positive number M, and $\epsilon > 0$, define the retraction function Π_M as follows*

$$
\Pi_M(x) = \begin{cases} x & |x| \leq M \\ \frac{x}{|x|} M & \textit{otherwise} \end{cases}
$$

Let $T = \{x : |x| \geq M + \epsilon\}$. Then $Lip(\Pi_M|T) \leq \frac{M}{M+\epsilon}$.

Proof. For $x, y \in T$,

$$
\begin{aligned}
|\Pi_M(x) - \Pi_M(y)|^2 &= |\Pi_M(x)|^2 + |\Pi_M(y)|^2 - 2 < \Pi_M(x), \Pi_M(y) > \\
&= 2M^2 - 2M^2 \frac{< x, y >}{|x||y|} \\
&= \frac{M^2}{|x||y|}(2|x||y| - 2 < x, y >) \\
&\leq \frac{M^2}{|x||y|}(|x|^2 + |y|^2 - 2 < x, y >) \\
&= \frac{M^2}{|x||y|}|x-y|^2 \leq \frac{M^2}{(M+\epsilon)^2}|x-y|^2.
\end{aligned}
$$

□

Definition 4.1. *For a Q-valued function f, define*

$$
|f(x)| := Max\{|f_1(x)|, |f_2(x)|, \cdots, |f_Q(x)|\},
$$

where $f = \sum_{i=1}^{Q}[[f_i]]$.

Theorem 4.1. *If $f : \mathbb{B}_1^m(0) \to \mathbb{Q}$ is strictly defined and Dir minimizing with boundary data $g : \partial\mathbb{B}_1^m(0) \to \mathbb{Q}$, where $f \in \mathcal{Y}_2(\mathbb{B}_1^m(0), \mathbb{Q}), g \in \partial\mathcal{Y}_2(\partial\mathbb{B}_1^m(0), \mathbb{Q})$, then*

$$
\sup_{x \in \mathbb{B}_1^m(0)} |f(x)| \leq \sup_{x \in \partial\mathbb{B}_1^m(0)} |g(x)|.
$$

Proof. We may assume that $M := \sup_{x \in \partial \mathbb{B}_1^m(0)} |g(x)| < \infty$.
If the statement is not true, i.e. there is a point $x_0 \in \mathbb{B}_1^m(0)$, such that $|f(x_0)| > M$.
Claim: $f(x) = f(x_0)$ for all $x \in \mathbb{B}_1^m(0)$.
Define the set $S = \{x \in \mathbb{B}_1^m(0) : f(x) = f(x_0)\}$, which is not empty by the assumption. Since f is continuous from Theorem 2.4, S is closed in $\mathbb{B}_1^m(0)$.
Let Π_M be the retraction function from $\mathbb{R}^m$ to $\mathbb{B}_M^m(0)$. Consider the new comparison Q-valued function $h = (\Pi_M)_\sharp \circ f$, which has boundary data g and whose energy is no more than that of f because $\text{Lip}(\Pi_M) \leq 1$.
Take any point $y \in S$, because of the continuity of f, there is a neighborhood U of y in $\mathbb{B}_1^m(0)$ such that $|f(x)| \geq M + \epsilon, x \in U$, for some ϵ small enough.
From Lemma 4.1, we know that $\text{Lip}(\Pi_M|U) \leq \dfrac{M}{M+\epsilon}$, hence

$$\text{Dir}(h; U) \leq \frac{M}{M+\epsilon} \, \text{Dir}(f; U).$$

Therefore f must be constant in U(otherwise, its energy is nonzero. But the energy of h in U is strictly smaller than that of f, contradicting to the fact that f is Dir minimizing). So S is also open in $\mathbb{B}_1^m(0)$. Therefore, $S = \mathbb{B}_1^m(0)$, which is a contradiction to the assumption that $f|\partial \mathbb{B}_1^m(0) = g$. □

5 Hybrid Inequality

From now on, $m \geq 2$ and $n \geq 2$.

Lemma 5.1. *If* $u : \mathbb{B}_1^m(0) \to \underline{Q}(\mathbb{S}^{n-1})$ *is strictly defined and Dir minimizing, then for a.e.* $0 < r < 1$,

$$\int_{\partial \mathbb{B}_r^m(0)} |\frac{\partial u}{\partial r}|^2 dx \leq \int_{\partial \mathbb{B}_r^m(0)} |\nabla_{tan} u|^2 dx$$

Proof. For minimizing maps, we still have the squeeze formula:

$$r \cdot D'(r) = (m-2) \cdot D(r) + 2r \cdot \int_{\partial \mathbb{B}_r^m(0)} |\frac{\partial u}{\partial r}|^2 d\mathcal{H}^{m-1}.$$

Noticing that $D'(r) = \int_{\partial \mathbb{B}_r^m(0)} |\frac{\partial u}{\partial r}|^2 d\mathcal{H}^{m-1} + \int_{\partial \mathbb{B}_r^m(0)} |\nabla_{\tan} u|^2 d\mathcal{H}^{m-1}$, we have

$$2r \cdot \text{dir}(f, \partial \mathbb{B}_r^m(0)) = (m-2)D(r) + r \cdot D'(r)$$

Therefore, $r \cdot D'(r) \leq 2r \cdot \text{dir}(f, \partial \mathbb{B}_r^m(0))$, i.e. $D'(r) \leq 2\text{dir}(f, \partial \mathbb{B}_r^m(0))$.
Writing that in integration form,

$$\int_{\partial \mathbb{B}_r^m(0)} |\nabla_{\tan} u|^2 + \int_{\partial \mathbb{B}_r^m(0)} |\frac{\partial u}{\partial r}|^2 \leq 2 \int_{\partial \mathbb{B}_r^m(0)} |\nabla_{\tan} u|^2,$$

gives us the desired inequality. □

Theorem 5.1 (Hybrid Inequality). *There is a positive constant C, depending only on m, n, Q such that if $0 < \lambda < 1$, and if u is a Dir-minimizer in $\mathcal{Y}_2(\mathbb{B}_1^m(0), \underline{Q}(S^{n-1}))$, then*

$$E_{1/2}(u) \leq \lambda E_1(u) + C\lambda^{-1} \int_{\mathbb{B}_1^m(0)} |\xi \circ u - \overline{\xi \circ u}|^2 dx,$$

where $E_r(u) = r^{2-m} \int_{\mathbb{B}_r^m(0)} |Du|^2 dx$, $\overline{\xi \circ u} = \fint_{\mathbb{B}_1^m(0)} (\xi \circ u) dx$.

Proof. For an increasing function η on $[0, 1]$,

$$\{s : \eta'(s) \geq 8[\eta(1) - \eta(0)]\}$$

has Lebesgue measure $\leq 1/8$. In particular, there is an $r \in [1/2, 1]$ such that $u|\partial\mathbb{B}_r^m(0) \in \partial\mathcal{Y}_2(\partial\mathbb{B}_r^m(0), \underline{Q}(\mathbb{S}^{n-1}))$,

$$\int_{\partial\mathbb{B}_r^m(0)} |\nabla_{\tan} u|^2 d\mathcal{H}^{m-1} \leq 8 \int_{\mathbb{B}_1^m(0)} |Du|^2 d\mathcal{H}^m,$$

$$\int_{\partial\mathbb{B}_r^m(0)} |\xi \circ u - \overline{\xi \circ u}|^2 d\mathcal{H}^{m-1} \leq 8 \int_{\mathbb{B}_1^m(0)} |\xi \circ u - \overline{\xi \circ u}|^2 d\mathcal{H}^m.$$

We claim that there exists a map $w \in \mathcal{Y}_2(\mathbb{B}_r^m(0), \underline{Q}(\mathbb{S}^{n-1}))$ such that $w|\partial\mathbb{B}_r^m(0) = u|\partial\mathbb{B}_r^m(0)$ and

$$\int_{\mathbb{B}_r^m(0)} |Dw|^2 dx \leq K(\int_{\partial\mathbb{B}_r^m(0)} |\nabla_{\tan} u|^2 d\mathcal{H}^{m-1})^{1/2} (\int_{\partial\mathbb{B}_r^m(0)} |\xi \circ u - \overline{\xi \circ u}|^2 d\mathcal{H}^{m-1})^{1/2}$$

for some universal constant K. This is because, we choose $h : \mathbb{B}_r^m(0) \to \mathbb{Q}$ such that $h|\partial\mathbb{B}_r^m(0) = u|\partial\mathbb{B}_r^m(0)$ and h is Dir-minimizing.

$$\begin{aligned}
\int_{\mathbb{B}_r^m(0)} |Dh|^2 &= \int_{\partial\mathbb{B}_r^m(0)} < \xi \circ h(x), D(\xi \circ h)(x, \frac{x}{|x|}) > d\mathcal{H}^{m-1} \\
&= \int_{\partial\mathbb{B}_r^m(0)} < \xi \circ h(x) - \overline{\xi \circ u}, D(\xi \circ h)(x, \frac{x}{|x|}) > d\mathcal{H}^{m-1} \\
&\leq [\int_{\partial\mathbb{B}_r^m(0)} |\xi \circ h(x) - \overline{\xi \circ u}|^2 d\mathcal{H}^{m-1}]^{1/2} [\int_{\partial\mathbb{B}_r^m(0)} |\frac{\partial h}{\partial r}|^2 d\mathcal{H}^{m-1}]^{1/2} \\
&= [\int_{\partial\mathbb{B}_r^m(0)} |\xi \circ u(x) - \overline{\xi \circ u}|^2 d\mathcal{H}^{m-1}]^{1/2} [\int_{\partial\mathbb{B}_r^m(0)} |\frac{\partial h}{\partial r}|^2 d\mathcal{H}^{m-1}]^{1/2}
\end{aligned}$$

By Lemma 5.1, we have

$$\int_{\partial\mathbb{B}_r^m(0)} |\frac{\partial h}{\partial r}|^2 \leq \int_{\partial\mathbb{B}_r^m(0)} |\nabla_{\tan} h|^2 = \int_{\partial\mathbb{B}_r^m(0)} |\nabla_{\tan} u|^2$$

Therefore

$$\int_{\mathbb{B}_r^m(0)} |Dh|^2 \leq [\int_{\partial\mathbb{B}_r^m(0)} |\xi \circ u - \overline{\xi \circ u}|^2 d\mathcal{H}^{m-1}]^{1/2} [\int_{\partial\mathbb{B}_r^m(0)} |\nabla_{\tan} u|^2 d\mathcal{H}^{m-1}]^{1/2}$$

Unfortunately, the image of h may not lie in $\underline{Q}(\mathbb{S}^{n-1})$. To correct this, we consider, for $a \in \mathbb{B}^m_{1/2}(0)$, the projection

$$\Pi_a(x) = (x-a)/|x-a|,$$

and note that by Sard's Theorem, the composition $(\Pi_a)_\sharp \circ h \in \mathcal{Y}_2(\mathbb{B}^m_1(0), \underline{Q}(\mathbb{S}^{n-1}))$ for almost all a.

Using Fubini's Theorem, we estimate

$$\int_{\mathbb{B}^m_{1/2}(0)} \int_{\mathbb{B}^m_r(0)} |D((\Pi_a)_\sharp \circ h)(x)|^2 dxda$$

$$\leq 4 \int_{\mathbb{B}^m_r(0)} |Dh(x)|^2 \int_{\mathbb{B}^m_{1/2}(0)} (\mathcal{G}(h(x), Q[[a]]))^{-2} dadx$$

Now we claim that $\int_{\mathbb{B}^m_{1/2}(0)} (\mathcal{G}(h(x), Q[[a]]))^{-2} da < K$ for some universal constant K independent of x.

Let $h(x) = \sum_{i=1}^Q [[h_i(x)]]$, then $(\mathcal{G}(h(x), Q[[a]]))^2 = \sum_{i=1}^Q |h_i(x) - a|^2$. Hence

$$(\mathcal{G}(h(x), Q[[a]]))^{-2} \leq |h_i(x) - a|^{-2}, \text{ for any } i$$

Applying Theorem 4.1 to the function h, and noticing that

$$h|\partial\mathbb{B}^m_r(0) \in \partial\mathcal{Y}_2(\partial\mathbb{B}^m_r(0), \underline{Q}(\mathbb{S}^{n-1})),$$

we get

$$|h_i(x)| \leq 1, \text{ for any } i, \text{ any } x \in \mathbb{B}^m_r(0)$$

Therefore

$$\int_{\mathbb{B}^m_{1/2}(0)} (\mathcal{G}(h(x), Q[[a]]))^{-2} da \leq \int_{\mathbb{B}^m_{1/2}(0)} |h_i(x) - a|^{-2} da$$

$$= \int_{\mathbb{B}^m_{1/2}(-h_i(x))} |y|^{-2} dy, \text{ by changing of variable } a = h_i(x) + y$$

$$\leq \int_{\mathbb{B}^m_2(0)} |y|^{-2} dy < \infty.$$

Hence $\int_{\mathbb{B}^m_{1/2}(0)} \int_{\mathbb{B}^m_r(0)} |D((\Pi_a)_\sharp \circ h)(x)|^2 dxda \leq K \int_{\mathbb{B}^m_r(0)} |Dh(x)|^2 dx$ for some constant K. We may choose $a \in \mathbb{B}^m_{1/2}(0)$ such that $\int_{\mathbb{B}^m_r(0)} |D((\Pi_a)_\sharp \circ h)(x)|^2 dx \leq K \int_{\mathbb{B}^m_r(0)} |Dh(x)|^2 dx$. Letting $w = [(\Pi_a|\mathbb{S}^{n-1})^{-1}]_\sharp \circ (\Pi_a)_\sharp \circ h$, we conclude that $w|\partial\mathbb{B}^m_r(0) = u|\partial\mathbb{B}^m_r(0)$, and that

$$\int_{\mathbb{B}^m_r(0)} |Dw(x)|^2 dx \leq [\text{Lip}(\Pi_a|\mathbb{S}^{n-1})^{-1}]^2 \int_{\mathbb{B}^m_r(0)} |D((\Pi_a)_\sharp \circ h)|^2 dx$$

$$\leq K \int_{\mathbb{B}^m_r(0)} |Dh|^2 dx$$

Now back to our desired result,

$$\begin{aligned}
E_{1/2}(u) &= (1/2)^{2-m}\int_{\mathbb{B}^m_{1/2}(0)}|Du|^2dx\\
&\leq 2^{m-2}\int_{\mathbb{B}^m_r(0)}|Du|^2dx\\
&\leq 2^{m-2}\int_{\mathbb{B}^m_r(0)}|Dw|^2dx\\
&\leq 2^{m-2}K(\int_{\partial\mathbb{B}^m_r(0)}|\nabla_{\tan}u|^2d\mathcal{H}^{m-1})^{1/2}(\int_{\partial\mathbb{B}^m_r(0)}|\xi\circ u-\overline{\xi\circ u}|^2d\mathcal{H}^{m-1})^{1/2}
\end{aligned}$$

Applying the inequality $ab \leq \frac{1}{2}\delta a^2 + \frac{1}{2}\delta^{-1}b^2$, with $\delta = \frac{\lambda}{2^m K}$, we have

$$\begin{aligned}
E_{1/2}(u) &\leq 2^{m-2}K(\frac{1}{2}\delta\int_{\partial\mathbb{B}^m_r(0)}|\nabla_{\tan}u|^2 + \frac{1}{2}\delta^{-1}\int_{\partial\mathbb{B}^m_r(0)}|\xi\circ u-\overline{\xi\circ u}|^2)\\
&\leq 2^{m-2}K(4\delta\int_{\mathbb{B}^m_1(0)}|Du|^2 + 4\delta^{-1}\int_{\mathbb{B}^m_1(0)}|\xi\circ u-\overline{\xi\circ u}|^2)\\
&= 2^mK\delta\int_{\mathbb{B}^m_1(0)}|Du|^2 + 2^mK\delta^{-1}\int_{\mathbb{B}^m_1(0)}|\xi\circ u-\overline{\xi\circ u}|^2\\
&= \lambda E_1(u) + C\lambda^{-1}\int_{\mathbb{B}^m_1(0)}|\xi\circ u-\overline{\xi\circ u}|^2,
\end{aligned}$$

where $C = (2^mK)^2$. □

6 Energy Improvement

6.1 A Poincare-Type Theorem

Definition 6.1.

$$p^* = \begin{cases} \dfrac{mp}{m-p} & p < m \\ \textit{any real number} \in [1,\infty) & p = m \\ \infty & p > m \end{cases}$$

Theorem 6.1 ([ZW], Theorem 4.4.6). *Let $\Omega \subset \mathbb{R}^m$ be a bounded Lipschitz domain and suppose $u \in W^{1,p}(\Omega)$, $1 < p < \infty$. Let*

$$c(u) = \int_{\partial\Omega} u\,d\mathcal{H}^{m-1}$$

Then there is a constant $C = C(m,p,\Omega)$, such that

$$(\int_\Omega |u-c(u)|^{p*}dx)^{1/p^*} \leq C(\int_\Omega |Du|^p dx)^{1/p}.$$

Corollary 6.1. *Let L be a real positive number such that $\mathcal{H}^{m-1}(\partial \mathbb{B}^m_L(0)) = 1$. Suppose $u \in W^{1,2}(\mathbb{B}^m_L(0))$, and let*

$$c(u) = \int_{\partial \mathbb{B}^m_L(0)} u d\mathcal{H}^{m-1} = \fint_{\partial \mathbb{B}^m_L(0)} u d\mathcal{H}^{m-1}$$

Then there is a constant $C = C(m)$ such that

$$\int_{\mathbb{B}^m_L(0)} |u - c(u)|^2 dx \leq C \int_{\mathbb{B}^m_L(0)} |Du|^2 dx.$$

Proof. Case 1. $m > 2$:
By Hölder inequality, with parameters $m/(m-2)$ and $m/2$

$$\begin{aligned}
\int_{\mathbb{B}^m_L(0)} |u - c(u)|^2 dx &\leq (\int_{\mathbb{B}^m_L(0)} |u - c(u)|^{2\times\frac{m}{m-2}} dx)^{(m-2)/m} (\int_{\mathbb{B}^m_L(0)} 1^{\frac{m}{2}} dx)^{2/m} \\
&= (\mathcal{L}^m(\mathbb{B}^m_L(0)))^{2/m} (\int_{\mathbb{B}^m_L(0)} |u - c(u)|^{2^*} dx)^{(m-2)/m} \\
&\leq (\mathcal{L}^m(\mathbb{B}^m_L(0)))^{2/m} [C(\int_{\mathbb{B}^m_L(0)} |Du|^2 dx)^{1/2}]^{2^* \times (m-2)/m} \\
&= C \int_{\mathbb{B}^m_L(0)} |Du|^2 dx
\end{aligned}$$

Case 2. $m = 2$:
We just choose 2^* to be 2 in Theorem 6.1. ☐

6.2 Blowing-up Sequence

Definition 6.2.

$$\begin{aligned}
\mathcal{F} = \{&u \in \mathcal{Y}_2(\mathbb{B}^m_1(0); \underline{Q}(\mathbb{S}^{n-1})), \forall 0 < r < 1, \xi^{-1} \circ \rho \circ AV_{r,0}(\xi \circ u) = Q[[b_r]], \\
&b_r \in \mathbb{R}^n, \frac{1}{2} < |b_r| < \frac{3}{2}, \text{where } AV_{r,0}(\xi \circ u) = \fint_{\partial \mathbb{B}^m_r(0)} \xi \circ u\}
\end{aligned}$$

Theorem 6.2 (Energy Improvement). *There are positive constants ϵ and $0 < \theta < \frac{1}{2}$ such that if u is a Dir minimizer in $\mathcal{Y}_2(\mathbb{B}^m_1(0), \underline{Q}(\mathbb{S}^{n-1}))$, $u \in \mathcal{F}$, $E_1(u) < \epsilon^2$, then $E_\theta(u) \leq \theta^{\omega_{2.13}} E_1(u)$.*

Proof. Were the theorem false, there would be, for each $0 < \theta < 1/2$, a sequence $u_i \in \mathcal{F}$,$\epsilon_i^2 = E_1(u_i) \to 0$, but

$$E_\theta(u_i) > \theta^{\omega_{2.13}} \epsilon_i^2.$$

Let Π be the projection onto the unit sphere in $\mathbb{R}^n$, i.e. $\Pi(x) = \frac{x}{|x|}$. It is easy to check that when we restrict our attention to the set $U_\epsilon = \{x : 1-\epsilon < |x| < 1+\epsilon\}$,

the Lipschitz constant of Π is no more than $1/(1-\epsilon)$.
Define

$$\Pi_\sharp \circ \xi^{-1} \circ \rho \circ AV_{L,0}(\xi \circ u_i) = Q[[b_i]],$$

where L is defined in Corollary 6.1.
Consider the following blowing-up sequence

$$\frac{u_i - Q[[b_i]]}{\epsilon_i}.$$

The energy of each one is one by the definition of ϵ_i. As for their L^2 norms, we estimate as follows:

$$\begin{aligned}
\mathcal{G}(u_i, Q[[b_i]]) &= |\Pi_\sharp \circ \xi^{-1} \circ \rho \circ \xi \circ u_i - \Pi_\sharp \circ \xi^{-1} \circ \rho \circ \fint_{\partial \mathbb{B}_L^m(0)} \xi \circ u_i d\mathcal{H}^{m-1}| \\
&\leq (\mathrm{Lip}\Pi|U_{1/2})(\mathrm{Lip}\xi^{-1})(\mathrm{Lip}\rho)|\xi \circ u_i - \fint_{\partial \mathbb{B}_L^m(0)} \xi \circ u_i d\mathcal{H}^{m-1}| \\
&\leq 2(\mathrm{Lip}\xi^{-1})(\mathrm{Lip}\rho)|\xi \circ u_i - \fint_{\partial \mathbb{B}_L^m(0)} \xi \circ u_i d\mathcal{H}^{m-1}|
\end{aligned}$$

From Corollary 6.1, we have

$$\begin{aligned}
\int_{\mathbb{B}_L^m(0)} \mathcal{G}(u_i, Q[[b_i]])^2 dx &\leq C \int_{\mathbb{B}_L^m(0)} |\xi \circ u_i - \fint_{\partial \mathbb{B}_L^m(0)} \xi \circ u_i d\mathcal{H}^{m-1}|^2 dx \\
&\leq C \int_{\mathbb{B}_L^m(0)} |D(\xi \circ u_i)|^2 dx \\
&\leq C \int_{\mathbb{B}_1^m(0)} |Du_i|^2 dx
\end{aligned}$$

Hence the L^2-norm of the blowing-up sequence in $\mathbb{B}_L^m(0)$ is uniformly bounded. (Technically, we should therefore from now on, focus on $\mathbb{B}_L^m(0)$ instead of $\mathbb{B}_1^m(0)$. But since the regularity is only a local property, we may just stick to $\mathbb{B}_1^m(0)$ for convenience.)
We use Compactness Theorem 4.2 in [ZW1] to get a subsequence(for convenience, whenever we have to take a subsequence, we do not change the notation) such that

$$w_i := \frac{u_i - Q[[b_i]]}{\epsilon_i} \rightharpoonup w \text{ weakly in } \mathcal{Y}_2$$

$$w_i \to w \text{ strongly in } L^2$$

$$\int_{\mathbb{B}_1^m(0)} |Dw|^2 \leq \liminf_{k\to\infty} \int_{\mathbb{B}_1^m(0)} |Dw_i|^2 = 1,$$

for some $w \in \mathcal{Y}_2(\mathbb{B}_1^m(0), \mathbb{Q})$.

6.3 Blowing-up the Constraint

Since $\mathbb{S}^{n-1}$ is compact, we may assume that $b_i \to b \in \mathbb{S}^{n-1}$.
Let $u_i(x) = \sum_{j=1}^{Q} u_j^{(i)}(x)$, $w_i(x) = \sum_{j=1}^{Q} w_j^{(i)}(x)$. By definition we have

$$\frac{u_j^{(i)} - b_i}{\epsilon_i} = w_j^{(i)},$$

hence

$$\frac{u_j^{(i)}}{\epsilon_i} = \frac{b_i}{\epsilon_i} + w_j^{(i)}.$$

Take the norm of both sides,

$$\epsilon_i^{-2} = \epsilon_i^{-2} + |w_j^{(i)}|^2 + \frac{2}{\epsilon_i} < b_i, w_j^{(i)} >$$

Hence

$$< b_i, w_j^{(i)} >= -\frac{\epsilon_i}{2}|w_j^{(i)}|^2.$$

Let i go to infinity, we know $w \in \mathcal{Y}_2(\mathbb{B}_1^m(0), \underline{Q}(P))$ for some $n-1$ dimensional plane P passing through the origin and perpendicular to b.

6.4 Strong Convergence and Minimality

Now we want to show that w is Dir minimizing in $\mathcal{Y}_2(\mathbb{B}_1^m(0), \underline{Q}(P))$ and the convergence of w_i in $\mathcal{Y}_2$ is actually strong.
Let $\mathbb{B}_{\rho_0}^m(y) \subset \mathbb{B}_1^m(0)$, and let $\delta > 0$, $\theta \in (0,1)$ be given. Choose any $M \in \{1, 2, \cdots\}$ such that

$$\limsup_{i\to\infty} {\rho_0}^{2-m} \int_{\mathbb{B}_{\rho_0}^m(y)} |D(u_i/\epsilon_i)|^2 < M\delta$$

and note that if $\epsilon \in (0, (1-\theta)/M)$, we must have some integer $l \in \{1, 2, \cdots, M\}$ such that

$$\rho_0^{2-m} \int_{\mathbb{B}_{\rho_0(\theta+l\epsilon)}^m(y)\backslash\mathbb{B}_{\rho_0(\theta+(l-1)\epsilon)}^m(y)} |D(u_i/\epsilon_i)|^2 < \delta \text{ for infinitely many } i$$

This is because that otherwise we get that $\rho_0^{2-m} \int_{\mathbb{B}_{\rho_0}^m(y)} |D(u_i/\epsilon_i)|^2 \geq M\delta$ for all sufficiently large i by summation over l, contrary to the definition of M. Thus choosing such an l, letting $\rho = \rho_0(\theta + (l-1)\epsilon)$, and noting that $\rho(1+\epsilon) \leq \rho_0(\theta + l\epsilon) < \rho_0$, we get $\rho \in [\theta\rho_0, \rho_0)$ such that

$$\rho_0^{2-m} \int_{\mathbb{B}_{\rho(1+\epsilon)}^m(y)\backslash\mathbb{B}_{\rho}^m(y)} |D(u_i/\epsilon_i)|^2 < \delta \text{ for some subsequence } u_i.$$

By weak convergence, we have

$$\rho_0^{2-m} \int_{\mathbb{B}_{\rho(1+\epsilon)}^m(y)\backslash\mathbb{B}_{\rho}^m(y)} |D(w + \frac{Q[[b_i]]}{\epsilon_i})|^2 < \delta \text{ for some subsequence.}$$

We can not use the Luckhaus-type Theorem 3.2 in [ZW1] now, because $\epsilon_i w + Q[[b_i]] \notin \underline{Q}(\mathbb{S}^{n-1})$. But we can use the technique "$(\Pi_a|\mathbb{S}^{n-1})^{-1} \circ \Pi_a''$ as we did in proving the hybrid inequality to get a map denoted as

$$(\Pi_i)_\sharp \circ (\epsilon_i w + Q[[b_i]]) \in \mathcal{Y}_2(\mathbb{B}_1^m(0), \underline{Q}(\mathbb{S}^{n-1}))$$

such that

$$\rho_0^{2-m} \int_{\mathbb{B}^m_{\rho(1+\epsilon)}(y)\backslash \mathbb{B}^m_\rho(y)} |D((\Pi_i)_\sharp \circ (\epsilon_i w + Q[[b_i]]))|^2 \leq \text{ some constant} \cdot \delta$$

Now by Corollary 3.1(2) in [ZW1], since $\int_{\mathbb{B}^m_{\rho_0}(y)} \mathcal{G}(u_i, (\Pi_i)_\sharp \circ (\epsilon_i w + Q[[b_i]]))^2 \to 0$, for sufficiently large i, we can find $v_i \in \mathcal{Y}_2(\mathbb{B}^m_{\rho(1+\epsilon)}(y)\backslash \mathbb{B}^m_\rho(y); \underline{Q}(\mathbb{S}^{n-1}))$ such that $v_i = (\Pi_i)_\sharp \circ (\epsilon_i w + Q[[b_i]])$ in a neighborhood of $\partial \mathbb{B}^m_\rho(y)$, $v_i = u_i$ in a neighborhood of $\partial \mathbb{B}^m_{\rho(1+\epsilon)}(y)$ and

$$\begin{aligned}\rho^{2-m} \int_{\mathbb{B}^m_{\rho(1+\epsilon)}(y)\backslash \mathbb{B}^m_\rho(y)} |Dv_i|^2 \leq \\ C\rho^{2-m} \int_{\mathbb{B}^m_{\rho(1+\epsilon)}(y)\backslash \mathbb{B}^m_\rho(y)} (|D((\Pi_i)_\sharp \circ (\epsilon_i w + Q[[b_i]])|^2 \\ + |Du_i|^2 + \frac{\mathcal{G}(u_i, (\Pi_i)_\sharp \circ (\epsilon_i w + Q[[b_i]]))^2}{\epsilon^2 \rho^2}),\end{aligned}$$

where C depends only on m, n, Q.
Now let $v \in \mathcal{Y}_2(\mathbb{B}^m_{\theta\rho_0}(y), \underline{Q}(P))$ such that $v = w$ in a neighborhood of $\partial \mathbb{B}^m_{\theta\rho_0}(y)$. Define

$$\tilde{v} = (\Pi_i)_\sharp \circ (\epsilon_i v + Q[[b_i]]) \text{ in } \mathbb{B}^m_{\theta\rho_0}(y)$$
$$\tilde{v} = (\Pi_i)_\sharp \circ (\epsilon_i w + Q[[b_i]]) \text{ in } \mathbb{B}^m_{\rho_0}(y)\backslash \mathbb{B}^m_{\theta\rho_0}(y).$$

Let $\tilde{u_i}$ be defined by

$$\tilde{u_i} = \tilde{v} \text{ in } \mathbb{B}^m_\rho(y),$$
$$\tilde{u_i} = v_i \text{ in } \mathbb{B}^m_{(1+\epsilon)\rho}(y)\backslash \mathbb{B}^m_\rho(y),$$
$$\tilde{u_i} = u_i \text{ in } \mathbb{B}^m_{\rho_0}(y)\backslash \mathbb{B}^m_{(1+\epsilon)\rho}(y).$$

By the minimizing property of u_i, we have

$$\begin{aligned}\int_{\mathbb{B}^m_{(1+\epsilon)\rho}(y)} |Du_i|^2 &\leq \int_{\mathbb{B}^m_{(1+\epsilon)\rho}(y)} |D\tilde{u_i}|^2 \\ &= \int_{\mathbb{B}^m_\rho(y)} |D\tilde{v}|^2 + \int_{\mathbb{B}^m_{(1+\epsilon)\rho}(y)\backslash \mathbb{B}^m_\rho(y)} |Dv_i|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\rho^{2-m}\int_{\mathbb{B}^m_\rho(y)}|Dw|^2 &\le \liminf_{i\to\infty}\rho^{2-m}\int_{\mathbb{B}^m_\rho(y)}\frac{|Du_i|^2}{\epsilon_i^2}\\ &\le \liminf_{i\to\infty}\rho^{2-m}\int_{\mathbb{B}^m_\rho(y)}\frac{|D\tilde v|^2}{\epsilon_i^2}+\liminf_{i\to\infty}\rho^{2-m}\int_{\mathbb{B}^m_{(1+\epsilon)\rho}(y)\setminus\mathbb{B}^m_\rho(y)}\frac{|Dv_i|^2}{\epsilon_i^2}\\ &\le \rho^{2-m}\int_{\mathbb{B}^m_{\theta\rho_0}(y)}|Dv|^2+\rho^{2-m}\int_{\mathbb{B}^m_\rho(y)\setminus\mathbb{B}^m_{\theta\rho_0}(y)}|Dw|^2+C\delta\end{aligned}$$

Since δ was arbitrary, we have

$$\rho^{2-m}\int_{\mathbb{B}_{\theta\rho_0}(y)}|Dw|^2\le\rho^{2-m}\int_{\mathbb{B}_{\theta\rho_0}(y)}|Dv|^2$$

Therefore, w is minimizing on $\mathbb{B}^m_{\theta\rho_0}(y)$, and in view of the arbitrariness of θ and ρ_0, this shows that w is minimizing on all balls $\mathbb{B}^m_\rho(y)$ with $\mathbb{B}^m_\rho(y)\subset\mathbb{B}^m_1(0)$. Finally to prove that the convergence is strong we note that if we use $v=w$ as above, we can conclude

$$\liminf_{i\to\infty}\rho^{2-m}\int_{\mathbb{B}^m_\rho(y)}\frac{|Du_i|^2}{\epsilon_i^2}\le\rho^{2-m}\int_{\mathbb{B}^m_\rho(y)}|Dw|^2+C\delta$$

and hence, in view of the arbitrariness of θ and δ,

$$\rho^{2-m}\liminf_{i\to\infty}\int_{\mathbb{B}^m_{\rho_1}(y)}\frac{|Du_i|^2}{\epsilon_i^2}\le\rho^{2-m}\int_{\mathbb{B}^m_{\rho_0}(y)}|Dw|^2,$$

for each $\rho_1<\rho_0$. Evidently it follows from this(keeping in mind the arbitrariness of ρ_0) that

$$\liminf_{i\to\infty}\int_{\mathbb{B}^m_\rho(y)}\frac{|Du_i|^2}{\epsilon_i^2}\le\int_{\mathbb{B}^m_\rho(y)}|Dw|^2$$

for every ball $\mathbb{B}^m_\rho(y)$ such that $\mathbb{B}^m_\rho(y)\subset\mathbb{B}^m_1(0)$. Then since

$$\int_{\mathbb{B}^m_\rho(y)}|D(u_i/\epsilon_i)-Dw|^2=\int_{\mathbb{B}^m_\rho(y)}|Dw|^2+\int_{\mathbb{B}^m_\rho(y)}|D(u_i/\epsilon_i)|^2-2\int_{\mathbb{B}^m_\rho(y)}Dw\cdot D(u_i/\epsilon_i),$$

we can evidently select a subsequence which converges strongly to Dw on $\mathbb{B}^m_\rho(y)$. Since this holds for arbitrary $\mathbb{B}^m_\rho(y)\subset\mathbb{B}^m_1(0)$, it is then easy to see(by covering $\mathbb{B}^m_1(0)$ by a countable collection of balls $\mathbb{B}^m_{\rho_j}(y_j)$ with $\mathbb{B}^m_{\rho_j}(y_j)\subset\mathbb{B}^m_1(0)$) that there is a subsequence such that $D(u_i/\epsilon_i)$ converges strongly locally in all of $\mathbb{B}^m_1(0)$.

6.5 Proof of Energy Improvement

Let's estimate $\fint_{\mathbb{B}_r^m(0)} |\xi \circ u_i - \overline{\xi \circ u_i}|^2 dx$,where $\overline{\xi \circ u_i} = \fint_{\mathbb{B}_r^m(0)} \xi \circ u_i dx$.

$$\begin{aligned}
\fint_{\mathbb{B}_r^m(0)} |\xi \circ u_i - \overline{\xi \circ u_i}|^2 dx &\leq Cr^{2-m} \int_{\mathbb{B}_r^m(0)} |D(\xi \circ u_i)|^2 dx \text{ (by Poincare inequality)} \\
&= Cr^{2-m} \int_{\mathbb{B}_r^m(0)} |Du_i|^2 dx \\
&= Cr^{2-m} \epsilon_i^2 \int_{\mathbb{B}_r^m(0)} |D(u_i/\epsilon_i)|^2 dx \\
&= Cr^{2-m} \epsilon_i^2 \int_{\mathbb{B}_r^m(0)} |Dw_i|^2 dx
\end{aligned}$$

We have already proved that Dw_i converges strongly to Dw in $\mathcal{Y}_2$, hence

$$\fint_{\mathbb{B}_r^m(0)} |\xi \circ u_i - \overline{\xi \circ u_i}|^2 dx \leq Cr^{2-m} \epsilon_i^2 \int_{\mathbb{B}_r^m(0)} |Dw|^2 dx.$$

We also have proved the Dir minimality of w, hence by Theorem 2.4

$$\fint_{\mathbb{B}_r^m(0)} |\xi \circ u_i - \overline{\xi \circ u_i}|^2 dx \leq Cr^{2-m} \epsilon_i^2 r^{m-2+2\omega_{2.13}} = Cr^{2\omega_{2.13}} \epsilon_i^2.$$

Applying the Hybrid Inequality to $u_i(2\theta x)$, we get

$$E_{1/2}(u_i(2\theta x)) \leq \lambda E_1(u_i(2\theta x)) + C\lambda^{-1} \int_{\mathbb{B}_1^m(0)} |\xi \circ u_i(2\theta x) - \overline{\xi \circ u_i(2\theta x)}|^2 dx$$

which can be simplified to

$$\begin{aligned}
E_\theta(u_i) &\leq \lambda E_{2\theta}(u_i) + C\lambda^{-1} \fint_{\mathbb{B}_{2\theta}^m(0)} |\xi \circ u_i - \overline{\xi \circ u_i}|^2 dx \\
&\leq \lambda E_{2\theta}(u_i) + C\lambda^{-1} \cdot (2\theta)^{2\omega_{2.13}} \epsilon_i^2
\end{aligned}$$

Choosing the positive integer $k = k(\theta)$ for which $1/2 \leq 2^k\theta \leq 1$, we iterate $k-1$ more times to obtain

$$\begin{aligned}
E_\theta(u_i) &\leq \lambda^k E_{2^k\theta}(u_i) + \sum_{j=1}^{k} \lambda^{j-1} C\lambda^{-1} \fint_{\mathbb{B}_{2^j\theta}^m(0)} |\xi \circ u_i - \overline{\xi \circ u_i}|^2 \\
&\leq \lambda^k (1/2)^{2-m} \epsilon_i^2 + \sum_{j=1}^{k} \lambda^{j-1} C\lambda^{-1} C(2^j\theta)^{2\omega_{2.13}} \epsilon_i^2 \\
&\leq \lambda^k \cdot 2^{m-2} \epsilon_i^2 + \sum_{j=1}^{\infty} (\lambda \cdot 2^{2\omega_{2.13}})^j C\lambda^{-2} \theta^{2\omega_{2.13}} \epsilon_i^2 \\
&\leq [\lambda^k \cdot 2^{m-2} + \frac{\lambda \cdot 2^{2\omega_{2.13}}}{1 - \lambda \cdot 2^{2\omega_{2.13}}} C\lambda^{-2} \theta^{2\omega_{2.13}}] \epsilon_i^2
\end{aligned}$$

Take $\lambda = \theta^{\frac{m+\omega_{2.13}}{k}}$, we have $\lambda^k \cdot 2^{m-2} = \theta^{m+\omega_{2.13}} \cdot 2^{m-2} = \theta^m \cdot 2^{m-2} \cdot \theta^{\omega_{2.13}} \leq (1/2)^m \cdot 2^{m-2}\theta^{\omega_{2.13}} \leq \theta^{\omega_{2.13}}/4$.
Since $\lambda = \theta^{\frac{m+\omega_{2.13}}{k}} \leq (2^{-k})^{\frac{m+\omega_{2.13}}{k}} = 2^{-(m+\omega_{2.13})}$,

$$\begin{aligned}\frac{\lambda \cdot 2^{2\omega_{2.13}}}{1-\lambda \cdot 2^{2\omega_{2.13}}} C\lambda^{-2}\theta^{2\omega_{2.13}} &\leq \frac{2^{2\omega_{2.13}}C}{1-2^{\omega_{2.13}-m}}\theta^{-\frac{m+\omega_{2.13}}{k}}\theta^{2\omega_{2.13}} \\ &= K\theta^{\omega_{2.13}-\frac{m+\omega_{2.13}}{k}}\theta^{\omega_{2.13}}\end{aligned}$$

Let's choose θ small enough such that $\theta^{\omega_{2.13}-\frac{m+\omega_{2.13}}{k}} \leq 1/4K$. This is possible because it is equivalent to

$$\theta^{\omega_{2.13}} \leq \theta^{\frac{m+\omega_{2.13}}{k}}/4K$$

Noting that $\theta \geq 2^{-1-k}$, the right side of above one is greater than

$$2^{-(k+1)(m+\omega_{2.13})/k}/4K$$

which is bounded from below although when θ goes to zero, k goes to infinity. Thus for i sufficiently large enough,we have

$$E_\theta(u_i) \leq (\frac{1}{4}\theta^{\omega_{2.13}} + \frac{1}{4}\theta^{\omega_{2.13}})\epsilon_i^2 < \theta^{\omega_{2.13}}\epsilon_i^2,$$

contradicting the choice of u_i. $\square$

7 Energy Decay

Theorem 7.1 (Energy decay). *If $u \in \mathcal{F}$ is Dir minimizing, $\mathbb{B}^m_R(0) \subset \mathbb{B}^m_1(0)$, and $R^{2-m}\int_{\mathbb{B}^m_R(0)} |Du|^2 \leq \epsilon^2$, then*

$$\int_{\mathbb{B}^m_r(0)} |Du|^2 \leq \theta^{2-m-\omega_{2.13}} R^{-\omega_{2.13}}\epsilon^2 r^{m-2+\omega_{2.13}}, \text{ for } 0 \leq r \leq R$$

where ϵ and θ are as in the Energy Improvement.

Proof. Let $u_{\theta^i R} \equiv u(\theta^i Rx), i = 0, 1, \cdots$. It is easy to check

$$E_1(u_{\theta^k R}) = (\theta^k R)^{2-m} Dir(u, \mathbb{B}^m_{\theta^k R}(0)) = E_\theta(u_{\theta^{k-1}R})$$

Claim: $E_\theta(u_R) \leq \theta^{\omega_{2.13}}\epsilon^2$.
This is because $u_R \in \mathcal{F}$,and $E_1(u_R) = R^{2-m}Dir(u, \mathbb{B}^m_R(0)) \leq \epsilon^2$ by our assumption. Hence we can use the energy improvement to the function u_R to get that.
Claim: $E_\theta(u_{\theta R}) \leq \theta^{2\omega_{2.13}}\epsilon^2$.
Obviously, $u_{\theta R} \in \mathcal{F}$, moreover,

$$E_1(u_{\theta R}) = E_\theta(u_R) \leq \theta^{\omega_{2.13}} E_1(u_R) \leq \theta^{\omega_{2.13}}\epsilon^2 \leq \epsilon^2.$$

Hence using the energy improvement to function $u_{\theta R}$, we get

$$E_\theta(u_{\theta R}) \le \theta^{\omega_{2.13}} E_1(u_{\theta R}) \le \theta^{2\omega_{2.13}} \epsilon^2.$$

Continuing the process, we get

$$\begin{aligned} E_1(u_{\theta^k R}) = E_\theta(u_{\theta^{k-1} R}) &\le \theta^{\omega_{2.13}} E_1(u_{\theta^{k-1} R}) = \theta^{\omega_{2.13}} E_\theta(u_{\theta^{k-2} R}) \\ &\le \theta^{2\omega_{2.13}} E_1(u_{\theta^{k-2} R}) \cdots = \theta^{k\omega_{2.13}} \epsilon^2, \end{aligned}$$

for $k = 1, 2, 3, \cdots$.
Given $0 < r \le R$, choose k such that $\theta^{k+1} R < r \le \theta^k R$.

$$\begin{aligned} r^{2-m} \int_{\mathbb{B}^m_r(0)} |Du|^2 &\le (\theta^{k+1} R)^{2-m} \int_{\mathbb{B}^m_{\theta^k R}(0)} |Du|^2 \\ &= \theta^{2-m} (\theta^k R)^{2-m} \int_{\mathbb{B}^m_{\theta^k R}(0)} |Du|^2 \\ &= \theta^{2-m} E_1(u_{\theta^k R}) \\ &\le \theta^{2-m} \theta^{k\omega_{2.13}} \epsilon^2 \\ &= \theta^{2-m-\omega_{2.13}} \theta^{(k+1)\omega_{2.13}} \epsilon^2 \\ &\le \theta^{2-m-\omega_{2.13}} (r/R)^{\omega_{2.13}} \epsilon^2 \end{aligned}$$

□

8 $\mathcal{H}^{m-2}$(singular set) $= 0$

Theorem 8.1. *Let $u \in \mathcal{Y}_2(\mathbb{B}^m_1(0), \underline{Q}(\mathbb{S}^{n-1}))$ be a strictly defined, Dirichlet minimizing map. Then it is Hölder continuous away from the boundary except for a closed subset $S \subset \mathbb{B}^m_1(0)$ such that $\mathcal{H}^{m-2}(S) = 0$.*

Proof. Let

$$S = \{x \in \mathbb{B}^m_1(0), \limsup_{\rho \downarrow 0} \rho^{2-m} \int_{\mathbb{B}^m_\rho(x)} |Du|^2 > 0\}.$$

Obviously, S is closed and $\mathcal{H}^{m-2}(S) = 0$ (see for example Lemma 2.1.1 in [LY]).
Let's look at a point $a \in \mathbb{B}^m_1(0) \sim S$. We may assume $a = 0$.
Let ϵ be the constant in the Energy Improvement, and $k = k(Q, m, n)$ be the constant in the "small energy regularity" theorem in [LC1]. Since $0 \notin S$, there is $R > 0$ such that $\mathbb{B}^m_{2R}(0) \subset \mathbb{B}^m_1(0) \sim S$ and

$$R^{2-m} \int_{\mathbb{B}^m_{2R}(0)} |Du|^2 \le \min\{\epsilon^2, k\}.$$

For any $b \in \mathbb{B}^m_R(0)$,

$$R^{2-m} \int_{\mathbb{B}^m_R(b)} |Du|^2 \le R^{2-m} \int_{\mathbb{B}^m_{2R}(0)} |Du|^2 \le \min\{\epsilon^2, k\}.$$

We have two possibilities:
Case 1: $b \notin B_0$. By the "small energy regularity" theorem in [LC1], we have

$$\int_{\mathbb{B}^m_r(b)} |Du|^2 \leq \text{some constant} \cdot r^{m-2+\beta}, 0 \leq r \leq R$$

where β is the constant given in [LC1].
Case 2: $b \in B_0$. From Energy Decay we have

$$\int_{\mathbb{B}^m_r(b)} |Du|^2 \leq \theta^{2-m-\omega_{2.13}} R^{-\omega_{2.13}} \epsilon^2 r^{m-2+\omega_{2.13}}, 0 \leq r \leq R.$$

Therefore $u \in C^{0,\min\{\omega_{2.13},\beta\}/2}[\mathbb{B}^m_R(0)]$ by Morrey's growth lemma. $\square$

9 Dimension Reduction

9.1 Monotonicity Formula

Suppose $u : \mathbb{B}^m_1(0) \to \underline{Q}(\mathbb{S}^{n-1}) \subset \underline{Q}(\mathbb{R}^n)$ is Dir minimizing, although $\xi \circ u$ is not necessarily harmonic, we still have the following results:
Consider the domain variation

$$u_s(x) = u(x + s\zeta(x)), \text{ where } \zeta = (\zeta^1, \cdots, \zeta^m), \text{ with } \zeta^j \in C^\infty_c(\mathbb{B}^m_1(0)).$$

We should have

$$\frac{d}{ds}|_{s=0} \text{ Energy of } \xi \circ u_s = 0.$$

If we let $f = \xi \circ u$, it is easy to check as in [SL], §2.2

$$\int_{\mathbb{B}^m_1(0)} \sum_{i,j=1}^m (|Df|^2 \delta_{ij} - 2D_i f \cdot D_j f) D_i \zeta^j = 0.$$

Theorem 9.1 (Monotonicity Formula). *If $u : \mathbb{B}^m_1(0) \to \underline{Q}(\mathbb{S}^{n-1}) \subset \underline{Q}(\mathbb{R}^n)$ is Dir minimizing, then*

$$\rho^{2-m} \int_{\mathbb{B}^m_\rho(x)} |Du|^2 - \sigma^{2-m} \int_{\mathbb{B}^m_\sigma(x)} |Du|^2 = 2 \int_{\mathbb{B}^m_\rho(x) \setminus \mathbb{B}^m_\sigma(x)} R^{2-m} |\frac{\partial u}{\partial R}|^2$$

for any $0 < \sigma < \rho < \rho_0$, provided $\mathbb{B}^m_{\rho_0}(x) \subset \mathbb{B}^m_1(0)$, where $R = |y - x|$ and $\partial/\partial R$ means directional derivative in the radial direction $|y - x|^{-1}(y - x)$.

Proof. Just apply the argument of ([SL], §2.4) to the function $\xi \circ u$ to get

$$\rho^{2-m} \int_{\mathbb{B}^m_\rho(x)} |D(\xi \circ u)|^2 - \sigma^{2-m} \int_{\mathbb{B}^m_\sigma(x)} |D(\xi \circ u)|^2 = 2 \int_{\mathbb{B}^m_\rho(x) \setminus \mathbb{B}^m_\sigma(x)} R^{2-m} |\frac{\partial(\xi \circ u)}{\partial R}|^2$$

and notice that $|D_v(\xi \circ u)| = |D_v u|$. $\square$

Remark 9.1. *(1) From above, $\rho^{2-m}\int_{\mathbb{B}^m_\rho(x)}|Du|^2$ is an increasing function of ρ for $\rho\in(0,\rho_0)$, and hence the limit as $\rho\to 0$ of $\rho^{2-m}\int_{\mathbb{B}^m_\rho(x)}|Du|^2$ exists; this limit is denoted as $\Theta_u(x)$. It is also easy to see that the density Θ_u is upper semi-continuous on $\mathbb{B}^m_1(0)$.*
(2) Another important additional conclusion, which we see by taking the limit as $\sigma\to 0$ in the monotonicity formula, is that $\int_{\mathbb{B}^m_\rho(x)}R^{2-m}|\frac{\partial u}{\partial R}|^2<\infty$ and

$$\rho^{2-m}\int_{\mathbb{B}^m_\rho(x)}|Du|^2-\Theta_u(x)=2\int_{\mathbb{B}^m_\rho(x)}R^{2-m}|\frac{\partial u}{\partial R}|^2.$$

9.2 Definition of Tangent Maps

Let $\mathbb{B}^m_{\rho_0}(y)$ with $\mathbb{B}^m_{\rho_0}(y)\subset\mathbb{B}^m_1(0)$, and for any $\rho>0$ consider the scaled function $u_{y,\rho}$ defined by

$$u_{y,\rho}(x)=u(y+\rho x).$$

If $\sigma>0$ is arbitrary and $\rho<\rho_0/\sigma$, we have (using $Du_{y,\rho}(x)=\rho(Du)(y+\rho x)$, and making a change of variable $\tilde{x}=y+\rho x$ in the energy integral of $u_{y,\rho}$)

$$\sigma^{2-m}\int_{\mathbb{B}^m_\sigma(0)}|Du_{y,\rho}|^2=(\sigma\rho)^{2-m}\int_{\mathbb{B}^m_{\sigma\rho}(y)}|Du|^2\le{\rho_0}^{2-m}\int_{\mathbb{B}^m_{\rho_0}(y)}|Du|^2 \qquad (1)$$

Thus if $\rho_j\downarrow 0$ then $\limsup_{j\to\infty}\int_{\mathbb{B}^m_\sigma(0)}|Du_{y,\rho_j}|^2<\infty$ for each $\sigma>0$. Their L^2-norms

$$\int_{\mathbb{B}^m_\sigma(0)}|u_{y,\rho}|^2=\rho^{-m}\int_{\mathbb{B}^m_{\sigma\rho}(y)}|u|^2<\infty$$

uniformly for ρ because $u(x)\in\underline{Q}(\mathbb{S}^{n-1})$.
So we can use Compactness Theorem 4.3 in [ZW1] to get a subsequence $\rho_{j'}$ such that $u_{y,\rho_{j'}}\to\varphi$ locally in $\mathbb{R}^m$ with respect to the $\mathcal{Y}_2-$norm, where $\varphi:\mathbb{R}^m\to\underline{Q}(\mathbb{S}^{n-1})$ is an energy minimizing map, called a tangent map of u at y.

9.3 Properties of Tangent Maps

Let $\rho_j\downarrow 0$ be one of the sequences such that the re-scaled maps $u_{y,\rho_j}\to\varphi$ as described above. Since u_{y,ρ_j} converges in energy to φ, we have, after setting $\rho=\rho_j$ and taking limits on each side of (1) as $j\to\infty$,

$$\sigma^{2-m}\int_{\mathbb{B}^m_\sigma(0)}|D\varphi|^2=\Theta_u(y).$$

Thus in particular, $\sigma^{2-m}\int_{\mathbb{B}^m_\sigma(0)}|D\varphi|^2$ is a constant function of σ and since by definition $\Theta_\varphi(0)=\lim_{\sigma\downarrow 0}\sigma^{2-m}\int_{\mathbb{B}^m_\sigma(0)}|D\varphi|^2$, we have

$$\Theta_u(y)=\Theta_\varphi(0)\equiv\sigma^{2-m}\int_{\mathbb{B}^m_\sigma(0)}|D\varphi|^2,\forall\sigma>0 \qquad (2)$$

Thus any tangent map of u at y has constant scaled energy and equal to the density of u at y.
Furthermore, we apply the monotonicity formula to φ to get

$$0 = \sigma^{2-m} \int_{\mathbb{B}^m_\sigma(0)} |D\varphi|^2 - \tau^{2-m} \int_{\mathbb{B}^m_\tau(0)} |D\varphi|^2 = \int_{\mathbb{B}^m_\sigma(0) \setminus \mathbb{B}^m_\tau(0)} R^{2-m} |\frac{\partial \varphi}{\partial R}|^2.$$

So that $\partial\varphi/\partial R = 0$ a.e, and since $\varphi \in \mathcal{Y}_2(\mathbb{R}^m, \underline{Q}(\mathbb{S}^{n-1}))$ it is correct to conclude from this, by integration along rays, that

$$\varphi(\lambda x) \equiv \varphi(x)\ \forall \lambda > 0, x \in \mathbb{R}^m$$

Theorem 9.2. *$y \in$ reg $u \Leftrightarrow \Theta_u(y) = 0 \Leftrightarrow \exists$ a constant tangent map φ of u at y*

Proof. The first part of the statement is easily obtained from Theorem 8.1. The second part comes from (2). □

9.4 Properties of Homogeneous Degree Zero Minimizers

Suppose $\varphi : \mathbb{R}^m \to \underline{Q}(\mathbb{S}^{n-1})$ is a homogeneous degree zero minimizer. We first observe that the density $\Theta_\varphi(y)$ is maximum at $y = 0$; in fact, by the monotonicity formula, for each $\rho > 0$ and each $y \in \mathbb{R}^m$

$$2 \int_{\mathbb{B}^m_\rho(y)} R_y^{2-m} |\frac{\partial \varphi}{\partial R_y}|^2 + \Theta_\varphi(y) = \rho^{2-m} \int_{\mathbb{B}^m_\rho(y)} |D\varphi|^2,$$

where $R_y(x) \equiv |x - y|$ and $\partial/\partial R_y = |x-y|^{-1}(x-y) \cdot D$. Now $\mathbb{B}^m_\rho(y) {}^{\subset} \mathbb{B}^m_{\rho+|y|}(0)$, so that

$$\begin{aligned}
\rho^{2-m} \int_{\mathbb{B}^m_\rho(y)} |D\varphi|^2 &\leq \rho^{2-m} \int_{\mathbb{B}^m_{\rho+|y|}(0)} |D\varphi|^2 \\
&= (1 + \frac{|y|}{\rho})^{m-2} (\rho + |y|)^{2-m} \int_{\mathbb{B}^m_{\rho+|y|}(0)} |D\varphi|^2 \\
&\equiv (1 + \frac{|y|}{\rho})^{m-2} \Theta_\varphi(0)
\end{aligned}$$

Thus letting $\rho \uparrow \infty$, we get

$$2 \int_{\mathbb{R}^m} R_y^{2-m} |\frac{\partial \varphi}{\partial R_y}|^2 + \Theta_\varphi(y) \leq \Theta_\varphi(0),$$

which establishes the required inequality

$$\Theta_\varphi(y) \leq \Theta_\varphi(0).$$

Notice also that this argument shows that the equality implies that $\partial\varphi/\partial R_y = 0$ a.e; that is $\varphi(y + \lambda x) \equiv \varphi(y + x)$ for each $\lambda > 0$. Since we also have $\varphi(\lambda x) \equiv \varphi(x)$ we can then compute that for any $\lambda > 0$ and $x \in \mathbb{R}^m$ that

$$\begin{aligned}
\varphi(x) = \varphi(\lambda x) = \varphi(y + (\lambda x - y)) &= \varphi(y + \lambda^{-2}(\lambda x - y)) \\
= \varphi(\lambda(y + \lambda^{-2}(\lambda x - y))) &= \varphi(x + ty),
\end{aligned}$$

where $t = \lambda - \lambda^{-1}$ is an arbitrary real number. So let $S(\varphi)$ be defined by

$$S(\varphi) = \{y \in \mathbb{R}^m : \Theta_\varphi(y) = \Theta_\varphi(0)\}.$$

Then we have shown that $\varphi(x) \equiv \varphi(x + ty)$ for all $x \in \mathbb{R}^m$,$t \in \mathbb{R}$, and $y \in S(\varphi)$.Then of course $\varphi(x + az_1 + bz_2) \equiv \varphi(x)$ for all $a, b \in \mathbb{R}$ and $z_1, z_2 \in S(\varphi)$. But if $z \in \mathbb{R}^m$ and $\varphi(x + z) \equiv \varphi(x)$ for all $x \in \mathbb{R}^m$, then trivially, $\Theta_\varphi(z) = \Theta_\varphi(0)$(and hence $z \in S(\varphi)$ by definition of $S(\varphi)$), so we conclude

$$S(\varphi) \text{ is a linear subspace of } \mathbb{R}^m \text{ and } \varphi(x + y) \equiv \varphi(x), x \in \mathbb{R}^m, y \in S(\varphi).$$

(Thus φ is invariant under the composition with translation by elements of $S(\varphi)$.) Notice of course that

$$\dim S(\varphi) = n \Leftrightarrow S(\varphi) = \mathbb{R}^m \Leftrightarrow \varphi = \text{ const.}$$

Also, a homogeneous degree zero map which is not constant clearly can not be continuous at 0, so we always have $0 \in$ sing φ if φ is non-constant, and hence, since $\varphi(x + z) \equiv \varphi(x)$ for any $z \in S(\varphi)$, we have

$$S(\varphi) \subset \text{ sing}\varphi$$

for any non-constant homogeneous degree zero minimizer φ.

9.5 Further Properties of sing u

We know

$$y \in \text{ sing } u \Leftrightarrow \dim S(\varphi) \leq n - 1 \text{ for every tangent map } \varphi \text{ of } u \text{ at } y \qquad (3)$$

Now for each $j = 0, 1, \cdots, n - 1$ we define

$$S_j = \{y \in \text{ sing } u : \dim S(\varphi) \leq j \text{ for all tangent maps } \varphi \text{ of } u \text{ at } y\}$$

Then we have

$$S_0 \subset S_1 \subset \cdots \subset S_{m-3} = S_{m-2} = S_{m-1} = \text{ sing } u.$$

To see this first note that $S_{j-1} \subset S_j$ is true by definition and $S_{m-1} =$ sing u is just (3). Also, if S_{m-3} is not equal to both S_{m-2} and S_{m-1}, then we can find $y \in$ sing u at which there is a tangent map φ with $\dim S(\varphi) = m - 1$ or $m - 2$; but then $\mathcal{H}^{m-2}(S(\varphi)) = \infty$ and hence (since $S(\varphi) \subset$ sing φ) we have $\mathcal{H}^{m-2}($ sing $\varphi) = \infty$, contradicting the fact that $\mathcal{H}^{m-2}(\text{sing } \varphi) = 0$ by Theorem 8.1.

Lemma 9.1. *For each $j = 0, 1, \cdots, m - 3, \dim S_j \leq j$, and, for each $\alpha \geq 0$, $S_0 \cap \{x : \Theta_u(x) = \alpha\}$ is a discrete set.*

Proof. The proof is exactly the same as in [SL], §3.4 □

Corollary 9.1. *Let $u \in \mathcal{Y}_2(\mathbb{B}_1^m(0), \underline{Q}(\mathbb{S}^{n-1}))$ be a strictly defined, Dirichlet minimizing map. Then it is Hölder continuous away from the boundary except for a closed subset $S \subset \mathbb{B}_1^m(0)$ such that $\dim(S) \leq m - 3$.*

Proof. Combine Lemma 9.1 with the fact that $\text{sing}(u) = S_{m-3}$. □

References

[AF] Frederick J. Almgren, Jr. Q-valued functions minimizing dirichlet's integral and the regularity of the area-minimizing rectifiable currents up to codimension 2, World Scientific 2000

[CS] Sheldon Chang, Two dimensional area minimizing integral currents are classical minimal surfaces, J. of the Amer. Math. Soc. 1, 699-778, (1988)

[FH] Herbert Federer, Geometric measure theory, Springer-Verlag, New York, 1969

[GJ] Jordan Goblet, A selection theory for multiple-valued functions in the sense of Almgren, Annales Academiae Scientiarum Fennicae Mathematica, Vol. 31. 2006, 297-314

[HK] Robert Hardt, David Kinderlehrer, Some regularity results in ferromagnetism, Commun. in P.D.E, 25, 1235-1258 (2000)

[HKL] Robert Hardt, David Kinderlehrer and Fang-Hua Lin, Existence and partial regularity of static liquid crystal configurations, Commun. Math. Phys. 105, 547-570 (1986)

[HW] M.C.Hong, C.Y.Wang, On the singular set of stable-stationary harmonic maps, Calc. Var. Partial Differential Equations, 9 (1999), no. 2, 141-156

[LC1] Chun-Chi Lin, Variational problems with multiple-valued functions and mappings, PhD Thesis, Rice University

[LC2] Chun-Chi Lin, On the regularity of two-dimensional stationary-harmonic multiple-valued functions, preprint

[LW] Fang-Hua Lin, Changyou Wang, Stable-stationary harmonic maps to spheres, Acta Math. Sin. (Engl. Ser.) 22 (2006), no. 2, 319–330

[LY] Fang-Hua Lin, Xiaoping Yang, Geometric Measure Theory: An Introduction, International Press, 2003

[MP] Pertti Mattila, Lower semicontinuity, existence and regularity theorems for elliptic variational integrals of multiple valued functions, Transactions of the American Mathematical Society, Vol. 280, No. 2, 1983

[SL] Leon Simon, Theorems on regularity and singularity of energy minimizing maps, Birkh$\check{a}$user, 1996

[SU1] Richard Schoen, Karen Uhlenbeck, A regularity theory for harmonic maps, J. Diff. Geom. 17 (1982), 307-335; 18 (1983), 329

[SU2] Richard Schoen, Karen Uhlenbeck, Regularity of minimizing harmonic maps into the sphere, Invent. Math, 78, 89-100 (1984)

[ZW] William P. Ziemer, Weakly differentiable functions, Springer-Verlag, 1989

[ZW1] Wei Zhu, An energy reducing flow for multiple-valued functions, arXiv:math.AP/0606478

[ZW2] Wei Zhu, Analysis on metric space $\mathbb{Q}$, arXiv:math.GN/0607560

[ZW3] Wei Zhu, A theorem on frequency function for multiple-valued Dirichlet minimizing functions, arXiv:math.AP/0607576